

CONSECUTIVE PRIMES AND BEATTY SEQUENCES

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ABSTRACT. Fix irrational numbers $\alpha, \hat{\alpha} > 1$ of finite type and real numbers $\beta, \hat{\beta} \geq 0$, and let \mathcal{B} and $\hat{\mathcal{B}}$ be the Beatty sequences

$$\mathcal{B} := (\lfloor \alpha m + \beta \rfloor)_{m \in \mathbb{N}} \quad \text{and} \quad \hat{\mathcal{B}} := (\lfloor \hat{\alpha} m + \hat{\beta} \rfloor)_{m \in \mathbb{N}}.$$

In this note, we study the distribution of pairs $(p, p^\#)$ of consecutive primes for which $p \in \mathcal{B}$ and $p^\# \in \hat{\mathcal{B}}$. Under a strong (but widely accepted) form of the Hardy-Littlewood conjectures, we show that

$$|\{p \leq x : p \in \mathcal{B} \text{ and } p^\# \in \hat{\mathcal{B}}\}| = (\alpha \hat{\alpha})^{-1} \pi(x) + O(x(\log x)^{-3/2+\varepsilon}).$$

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1. INTRODUCTION

For any given real numbers $\alpha > 0$ and $\beta \geq 0$, the associated (generalized) Beatty sequence is defined by

$$\mathcal{B}_{\alpha, \beta} := (\lfloor \alpha m + \beta \rfloor)_{m \in \mathbb{N}},$$

where $\lfloor t \rfloor$ is the largest integer not exceeding t . If α is irrational, it follows from a classical exponential sum estimate of Vinogradov [7] that $\mathcal{B}_{\alpha, \beta}$ contains infinitely many prime numbers; in fact, one has

$$\#\{\text{prime } p \leq x : p \in \mathcal{B}_{\alpha, \beta}\} \sim \alpha^{-1} \pi(x) \quad (x \rightarrow \infty),$$

where $\pi(x)$ is the prime counting function.

Throughout this paper, we fix two (not necessarily distinct) irrational numbers $\alpha, \hat{\alpha} > 1$ and two (not necessarily distinct) real numbers $\beta, \hat{\beta} \geq 0$, and we denote

$$\mathcal{B} := \mathcal{B}_{\alpha, \beta} \quad \text{and} \quad \hat{\mathcal{B}} := \mathcal{B}_{\hat{\alpha}, \hat{\beta}}. \quad (1.1)$$

Our aim is to study the set of primes $p \in \mathcal{B}$ for which the next larger prime $p^\#$ lies in $\hat{\mathcal{B}}$. The results we obtain are conditional, relying only on the *Hardy-Littlewood conjectures* in the following strong form. Let \mathcal{H} be a finite subset of \mathbb{Z} , and let $\mathbf{1}_{\mathbb{P}}$ denote the indicator function of the primes. The Hardy-Littlewood conjecture for \mathcal{H} asserts that the estimate

$$\sum_{n \leq x} \prod_{h \in \mathcal{H}} \mathbf{1}_{\mathbb{P}}(n + h) = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{|\mathcal{H}|}} + O(x^{1/2+\varepsilon}) \quad (1.2)$$

holds for any fixed $\varepsilon > 0$, where $\mathfrak{S}(\mathcal{H})$ is the singular series given by

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{|(\mathcal{H} \bmod p)|}{p} \right) \left(1 - \frac{1}{p} \right)^{-|\mathcal{H}|}.$$

Our main result is the following.

THEOREM 1.1. *Fix irrational numbers $\alpha, \hat{\alpha} > 1$ of finite type and real numbers $\beta, \hat{\beta} \geq 0$, and let \mathcal{B} and $\hat{\mathcal{B}}$ be the Beatty sequences given by (1.1). For every prime p , let p^\sharp denote the next larger prime. Suppose that the Hardy-Littlewood conjecture (1.2) holds for every finite subset \mathcal{H} of \mathbb{Z} . Then, for any fixed $\varepsilon > 0$, the counting function*

$$\pi(x; \mathcal{B}, \hat{\mathcal{B}}) := |\{p \leq x : p \in \mathcal{B} \text{ and } p^\sharp \in \hat{\mathcal{B}}\}|$$

satisfies the estimate

$$\pi(x; \mathcal{B}, \hat{\mathcal{B}}) = (\alpha\hat{\alpha})^{-1}\pi(x) + O(x(\log x)^{-3/2+\varepsilon}),$$

where the implied constant depends only on $\alpha, \hat{\alpha}$ and ε .

Our results are largely inspired by the recent breakthrough paper of Lemke Oliver and Soundararajan [3], which studies the surprisingly erratic distribution of pairs of consecutive primes amongst the $\phi(q)^2$ permissible reduced residue classes modulo q . In [3] a conjectural explanation for this phenomenon is given which is based on the strong form of the Hardy-Littlewood conjectures considered in this note, that is, under the hypothesis that the estimate (1.2) holds for every finite subset \mathcal{H} of \mathbb{Z} .

2. PRELIMINARIES

2.1. Notation. The notation $\llbracket t \rrbracket$ is used to denote the distance from the real number t to the nearest integer; that is,

$$\llbracket t \rrbracket := \min_{n \in \mathbb{Z}} |t - n| \quad (t \in \mathbb{R}).$$

We denote by $[t]$ and $\{t\}$ the greatest integer $\leq t$ and the fractional part of t , respectively. We also write $\mathbf{e}(t) := e^{2\pi it}$ for all $t \in \mathbb{R}$, as usual.

Let \mathbb{P} denote the set of primes in \mathbb{N} . In what follows, the letter p always denotes a prime number, and p^\sharp is used to denote the smallest prime greater than p . In other words, p and p^\sharp are consecutive primes with $p^\sharp > p$. We also put

$$\delta_p := p^\sharp - p \quad (p \in \mathbb{P}).$$

For an arbitrary set \mathcal{S} , we use $\mathbf{1}_{\mathcal{S}}$ to denote its indicator function:

$$\mathbf{1}_{\mathcal{S}}(n) := \begin{cases} 1 & \text{if } n \in \mathcal{S}, \\ 0 & \text{if } n \notin \mathcal{S}. \end{cases}$$

Throughout the paper, implied constants in symbols O , \ll and \gg may depend (where obvious) on the parameters $\alpha, \hat{\alpha}, \varepsilon$ but are absolute otherwise. For given functions F and G , the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq c|G|$ holds with some constant $c > 0$.

2.2. Discrepancy. We recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $x_1, x_2, \dots, x_M \in [0, 1)$ is defined by

$$D(M) := \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \quad (2.1)$$

where the supremum is taken over all intervals $\mathcal{I} = (b, c)$ contained in $[0, 1)$, the quantity $V(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $x_m \in \mathcal{I}$, and $|\mathcal{I}| = c - b$ is the length of \mathcal{I} .

For any irrational number a we define its type $\tau = \tau(a)$ by the relation

$$\tau := \sup \left\{ t \in \mathbb{R} : \liminf_{n \rightarrow \infty} n^t \llbracket an \rrbracket = 0 \right\}.$$

Using Dirichlet's approximation theorem, one sees that $\tau \geq 1$ for every irrational number a . Thanks to the work of Khinchin [1] and Roth [5, 6] it is known that $\tau = 1$ for almost all real numbers (in the sense of the Lebesgue measure) and for all irrational algebraic numbers, respectively.

For a given irrational number a , it is well known that the sequence of fractional parts $\{a\}, \{2a\}, \{3a\}, \dots$, is uniformly distributed modulo one (see, for example, [2, Example 2.1, Chapter 1]). When a is of finite type, this statement can be made more precise. By [2, Theorem 3.2, Chapter 2] we have the following result.

LEMMA 2.1. *Let a be a fixed irrational number of finite type τ . For every $b \in \mathbb{R}$ the discrepancy $D_{a,b}(M)$ of the sequence of fractional parts $(\{am+b\})_{m=1}^M$ satisfies the bound*

$$D_{a,b}(M) \leq M^{-1/\tau+o(1)} \quad (M \rightarrow \infty),$$

where the function implied by $o(\cdot)$ depends only on a .

2.3. Indicator function of a Beatty sequence. As in §1 we fix (possibly equal) irrational numbers $\alpha, \hat{\alpha} > 1$ and (possibly equal) real numbers $\beta, \hat{\beta} \geq 0$, and we set

$$\mathcal{B} := \mathcal{B}_{\alpha,\beta} \quad \text{and} \quad \hat{\mathcal{B}} := \mathcal{B}_{\hat{\alpha},\hat{\beta}}.$$

In what follows we denote

$$a := \alpha^{-1}, \quad \hat{a} := \hat{\alpha}^{-1}, \quad b := \alpha^{-1}(1 - \beta) \quad \text{and} \quad \hat{b} := \hat{\alpha}^{-1}(1 - \hat{\beta}).$$

It is straightforward to show that

$$\mathbf{1}_{\mathcal{B}}(m) = \psi_a(am + b) \quad \text{and} \quad \mathbf{1}_{\hat{\mathcal{B}}}(m) = \psi_{\hat{a}}(\hat{a}m + \hat{b}) \quad (m \in \mathbb{N}), \quad (2.2)$$

where for any $t \in (0, 1)$ we use ψ_t to denote the periodic function of period one defined by

$$\psi_t(x) := \begin{cases} 1 & \text{if } 0 < \{x\} \leq t, \\ 0 & \text{if } t < \{x\} < 1 \text{ or } \{x\} = 0. \end{cases}$$

2.4. Modified Hardy-Littlewood conjecture. For their work on primes in short intervals, Montgomery and Soundararajan [4] have introduced the modified singular series

$$\mathfrak{S}_0(\mathcal{H}) := \sum_{\mathcal{T} \subseteq \mathcal{H}} (-1)^{|\mathcal{H} \setminus \mathcal{T}|} \mathfrak{S}(\mathcal{T}),$$

for which one has the relation

$$\mathfrak{S}(\mathcal{H}) = \sum_{\mathcal{T} \subseteq \mathcal{H}} \mathfrak{S}_0(\mathcal{T}).$$

Note that $\mathfrak{S}(\emptyset) = \mathfrak{S}_0(\emptyset) = 1$. The Hardy-Littlewood conjecture (1.2) can be reformulated in terms of the modified singular series as follows:

$$\sum_{n \leq x} \prod_{h \in \mathcal{H}} \left(\mathbf{1}_{\mathbb{P}}(n+h) - \frac{1}{\log n} \right) = \mathfrak{S}_0(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{|\mathcal{H}|}} + O(x^{1/2+\varepsilon}). \quad (2.3)$$

LEMMA 2.2. *We have*

$$\begin{aligned} \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{0, t\}) &\ll h^{1/2+\varepsilon}, \\ \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{t, h\}) &\ll h^{1/2+\varepsilon}, \\ \sum_{1 \leq t_1 < t_2 \leq h-1} \mathfrak{S}_0(\{t_1, t_2\}) &= -\frac{1}{2}h \log h + \frac{1}{2}Ah + O(h^{1/2+\varepsilon}), \end{aligned}$$

where $A := 2 - C_0 - \log 2\pi$ and C_0 denotes the Euler-Mascheroni constant.

Proof. Let us denote

$$B := \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{0, t\}), \quad C := \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{t, h\}),$$

and

$$D_{\pm} := \sum_{1 \leq t_1 < t_2 \leq h \pm 1} \mathfrak{S}_0(\{t_1, t_2\})$$

for either choice of the sign \pm . Clearly,

$$\mathfrak{S}_0(\{0, h\}) + B + C + D_- = D_+ \quad \text{and} \quad B = \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{0, h-t\}) = C.$$

From [4, Equation (16)] we derive the estimates

$$D_{\pm} = -\frac{1}{2}h \log h + \frac{1}{2}Ah + O(h^{1/2+\varepsilon}).$$

Using the trivial bound $\mathfrak{S}_0(\{0, h\}) \ll \log \log h$ and putting everything together, we finish the proof. \square

2.5. Technical lemmas. Let $\nu(u) := 1 - 1/\log u$. Note that $\nu(u) \asymp 1$ for $u \geq 3$.

LEMMA 2.3. *Let $c > 0$ be a constant, and suppose that f is a function such that $|f(h)| \leq h^c$ for all $h \geq 1$. Then, uniformly for $3 \leq u \leq x$ and $\lambda \in \mathbb{R}$ we have*

$$\sum_{\substack{h \leq (\log x)^3 \\ 2|h}} f(h) \nu(u)^h \mathbf{e}(\lambda h) = \sum_{\substack{h \geq 1 \\ 2|h}} f(h) \nu(u)^h \mathbf{e}(\lambda h) + O_c(x^{-1}).$$

Proof. Write $\nu(u)^h = e^{-h/H}$ with $H := -(\log \nu(u))^{-1}$. Since $H \leq \log u$ for $u \geq 3$, for any $h > (\log x)^3$ we have $h/H \geq h^{2/3}$ as $u \leq x$; therefore,

$$\left| \sum_{\substack{h > (\log x)^3 \\ 2|h}} f(h) \nu(u)^h \mathbf{e}(\lambda h) \right| \leq \sum_{h > (\log x)^3} h^c e^{-h^{2/3}} \leq x^{-1} \sum_{h > (\log x)^3} h^c e^{h^{1/3} - h^{2/3}} \ll_c x^{-1},$$

and the result follows. \square

The next statement is an analogue of [3, Proposition 2.1] and is proved using similar methods.

LEMMA 2.4. *Fix $\theta \in [0, 1]$ and $\vartheta = 0$ or 1 . For all $\lambda \in \mathbb{R}$ and $u \geq 3$, let*

$$R_{\theta, \vartheta; \lambda}(u) := \sum_{\substack{h \geq 1 \\ 2|h}} h^\theta (\log h)^\vartheta \nu(u)^h \mathbf{e}(\lambda h),$$

$$S_\lambda(u) := \sum_{\substack{h \geq 1 \\ 2|h}} \mathfrak{S}_0(\{0, h\}) \nu(u)^h \mathbf{e}(\lambda h).$$

When $\lambda = 0$ we have the estimates

$$R_{\theta, 0; 0}(u) = \frac{1}{2} \Gamma(1 + \theta) (\log u)^{1+\theta} + O(1),$$

$$R_{\theta, 1; 0}(u) = \frac{1}{2} (\log 2) \Gamma(1 + \theta) (\log u)^{1+\theta} + O(1),$$

$$S_0(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1).$$

On the other hand, if λ is such that $|\lambda| \geq (\log u)^{-1}$, then

$$\max \{|R_{\theta, \vartheta; \lambda}(u)|, |S_\lambda(u)|\} \ll \lambda^{-4}.$$

Proof. We adapt the proof of [3, Proposition 2.1]. As in Lemma 2.3 we write $\nu(u)^h = e^{-h/H}$ with $H := -(\log \nu(u))^{-1}$. We simplify the expressions $R_{\theta, \vartheta; \lambda}(u)$, $S_\lambda(u)$ and $T_\lambda(u)$ by writing

$$\nu(u)^h \mathbf{e}(\lambda h) = e^{-h/H_\lambda} \quad \text{with} \quad H_\lambda := \frac{H}{1 - 2\pi i \lambda H}.$$

Since $\Re(h/H_\lambda) = h/H > 0$ for any positive integer h , using the Cahen-Mellin integral we have

$$R_{\theta, \vartheta; \lambda}(u) = \sum_{\substack{h \geq 1 \\ 2|h}} h^\theta (\log h)^\vartheta e^{-h/H_\lambda} = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} \left(\sum_{\substack{h \geq 1 \\ 2|h}} \frac{h^\theta (\log h)^\vartheta}{h^s} \right) \Gamma(s) H_\lambda^s ds.$$

In particular,

$$R_{\theta, 0; \lambda}(u) = \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta(s - \theta) \Gamma(s) H_\lambda^s ds \quad (2.4)$$

and

$$R_{\theta, 1; \lambda}(u) = R_{\theta, 0; \lambda}(u) \log 2 - \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta'(s - \theta) \Gamma(s) H_\lambda^s ds. \quad (2.5)$$

When $\lambda \neq 0$ we have

$$\begin{aligned} |R_{\theta,0;\lambda}(u)| &\leq \frac{2^{\theta-4}|H_\lambda|^4}{2\pi} \int_{-\infty}^{\infty} |\zeta(4-\theta+it)\Gamma(4+it)| dt \\ &\ll |H_\lambda|^4 = \left(\frac{H^2}{1+4\pi^2\lambda^2 H^2} \right)^2, \end{aligned}$$

hence the bound $R_{\theta,0;\lambda}(u) \ll \lambda^{-4}$ holds if $|\lambda| \geq (\log u)^{-1}$ since $H \asymp \log u$ for $u \geq 3$. In the case that $\lambda = 0$, the stated estimate for $R_{\theta,0;0}(u)$ is obtained by shifting the line of integration in (2.4) to the line $\{\Re(s) = -\frac{1}{3}\}$ (say), taking into account the residues of the poles of the integrand at $s = 1 + \theta$ and $s = 0$.

Our estimates for $R_{\theta,1;\lambda}(u)$ are proved similarly, using (2.5) instead of (2.4) and taking into account that $\zeta'(s-\theta) = (s-1-\theta)^{-1} + O(1)$ for s near $1+\theta$.

Next, for all $\lambda \in \mathbb{R}$ and $u \geq 3$, let

$$T_\lambda(u) := \sum_{h \geq 1} \mathfrak{S}(\{0, h\}) e^{-h/H_\lambda}.$$

Since $\mathfrak{S}_0(\{0, h\}) = \mathfrak{S}(\{0, h\}) - 1$ for all integers h , and $\mathfrak{S}(\{0, h\}) = 0$ if h is odd, it follows that

$$S_\lambda(u) = T_\lambda(u) - R_{0,0;\lambda}(u) = T_\lambda(u) - \frac{1}{2} \log u + O(1).$$

Hence, to complete the proof of the lemma, it suffices to show that

$$T_0(u) = \log u - \frac{1}{2} \log \log u + O(1) \quad \text{and} \quad T_\lambda(u) \ll \lambda^{-4} \text{ if } |\lambda| \geq (\log u)^{-1}.$$

As in the proof of [3, Proposition 2.1], we consider the Dirichlet series

$$F(s) := \sum_{h \geq 1} \frac{\mathfrak{S}(\{0, h\})}{h^s},$$

which can be expressed in the form

$$F(s) = \frac{\zeta(s)\zeta(s+1)}{\zeta(2s+2)} \prod_p \left(1 - \frac{1}{(p-1)^2} + \frac{2p}{(p-1)^2(p^{s+1}+1)} \right),$$

and the final product is analytic for $\Re(s) > -1$. Using the Cahen-Mellin integral we have

$$T_\lambda(u) = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} F(s)\Gamma(s)H_\lambda^s ds. \quad (2.6)$$

For $\lambda \neq 0$ we have

$$|T_\lambda(u)| \leq \frac{|H_\lambda|^4}{2\pi} \int_{-\infty}^{\infty} |F(4+it)\Gamma(4+it)| dt \ll |H_\lambda|^4 = \left(\frac{H^2}{1+4\pi^2\lambda^2 H^2} \right)^2$$

hence $T_\lambda(u) \ll \lambda^{-4}$ holds provided that $|\lambda| \geq (\log u)^{-1}$. For $\lambda = 0$, we shift the line of integration in (2.6) to the line $\{\Re(s) = -\frac{1}{3}\}$ (say), taking into account the double pole at $s = 0$ and the simple pole at $s = 1$. This leads to the stated estimate for $T_0(u)$. \square

We also need the following integral estimate (proof omitted).

LEMMA 2.5. *For all $\lambda \in \mathbb{R}$ and $x \geq 3$, let*

$$I_\lambda(x) := \int_3^x \frac{\mathbf{e}(\lambda u)}{\nu(u) \log u} du.$$

When $\lambda = 0$ we have the estimate

$$I_0(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

whereas for any $\lambda \neq 0$ we have

$$I_\lambda(x) \ll |\lambda|^{-1}.$$

3. PROOF OF THEOREM 1.1

For every even integer $h \geq 2$ we denote

$$\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) := |\{p \leq x : p \in \mathcal{B}, p^\# \in \hat{\mathcal{B}} \text{ and } \delta_p = h\}| = \sum_{n \leq x} \mathbf{1}_{\mathcal{B}}(n) \mathbf{1}_{\hat{\mathcal{B}}}(n+h) f_h(n),$$

where

$$f_h(n) := \mathbf{1}_{\mathbb{P}}(n) \mathbf{1}_{\mathbb{P}}(n+h) \prod_{0 < t < h} (1 - \mathbf{1}_{\mathbb{P}}(n+t)) = \begin{cases} 1 & \text{if } n = p \in \mathbb{P} \text{ and } \delta_p = h, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\pi(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{\substack{h \leq (\log x)^3 \\ 2|h}} \pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) + O\left(\frac{x}{(\log x)^3}\right). \quad (3.1)$$

Fixing an even integer $h \in [1, (\log x)^3]$ for the moment, our initial goal is to express $\pi_h(x; \mathcal{B}, \hat{\mathcal{B}})$ in terms of the function

$$S_h(x) := \sum_{n \leq x} f_h(n)$$

recently introduced by Lemke Oliver and Soundararajan [3, Equation (2.5)]. In view of (2.2) we can write

$$\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{n \leq x} \psi_a(an+b) \psi_{\hat{a}}(\hat{a}(n+h) + \hat{b}) f_h(n). \quad (3.2)$$

According to a classical result of Vinogradov (see [8, Chapter I, Lemma 12]), for any Δ such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{a, 1-a\}$$

there is a real-valued function Ψ_a with the following properties:

- (i) Ψ_a is periodic with period one;
- (ii) $0 \leq \Psi_a(t) \leq 1$ for all $t \in \mathbb{R}$;
- (iii) $\Psi_a(t) = \psi_a(t)$ if $\Delta \leq \{t\} \leq a - \Delta$ or if $a + \Delta \leq \{t\} \leq 1 - \Delta$;
- (iv) Ψ_a is represented by a Fourier series

$$\Psi_a(t) = \sum_{k \in \mathbb{Z}} g_a(k) \mathbf{e}(kt),$$

where $g_a(0) = a$, and the Fourier coefficients satisfy the uniform bound

$$|g_a(k)| \ll \min\{|k|^{-1}, |k|^{-2} \Delta^{-1}\} \quad (k \neq 0). \quad (3.3)$$

For convenience, we denote

$$\mathcal{I}_a := [0, \Delta) \cup (a - \Delta, a + \Delta) \cup (1 - \Delta, 1),$$

so that $\Psi_a(t) = \psi_a(t)$ whenever $\{t\} \notin \mathcal{I}_a$. Defining $\Psi_{\hat{a}}$ and $\mathcal{I}_{\hat{a}}$ similarly with \hat{a} in place of a , and taking into account the properties (i)–(iii), from (3.2) we deduce that

$$\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{n \leq x} \Psi_a(an + b) \Psi_{\hat{a}}(\hat{a}(n + h) + \hat{b}) f_h(n) + O(V(x)), \quad (3.4)$$

where $V(x)$ is the number of positive integers $n \leq x$ for which

$$\{an + b\} \in \mathcal{I}_a \quad \text{or} \quad \{\hat{a}(n + h) + \hat{b}\} \in \mathcal{I}_{\hat{a}}.$$

Since \mathcal{I}_a and $\mathcal{I}_{\hat{a}}$ are unions of intervals with overall measure 4Δ , it follows from the definition (2.1) and Lemma 2.1 that

$$V(x) \ll \Delta x + x^{1-1/\tau+o(1)} \quad (x \rightarrow \infty). \quad (3.5)$$

Now let $K \geq \Delta^{-1}$ be a large real number, and let $\Psi_{a,K}$ be the trigonometric polynomial given by

$$\Psi_{a,K}(t) := \sum_{|k| \leq K} g_a(k) \mathbf{e}(kt).$$

Using (3.3) it is clear that the estimate

$$\Psi_a(t) = \Psi_{a,K}(t) + O(K^{-1}\Delta^{-1}) \quad (3.6)$$

holds uniformly for all $t \in \mathbb{R}$. Defining $\Psi_{\hat{a},K}$ in a similar way, combining (3.6) with (3.4), and taking into account (3.5), we derive the estimate

$$\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \Sigma_h + O(\Delta x + x^{1-1/\tau+\varepsilon} + K^{-1}\Delta^{-1}x),$$

where

$$\begin{aligned} \Sigma_h &:= \sum_{n \leq x} \Psi_{a,K}(an + b) \Psi_{\hat{a},K}(\hat{a}(n + h) + \hat{b}) f_h(n) \\ &= \sum_{n \leq x} \sum_{|k|, |\ell| \leq K} g_a(k) \mathbf{e}(k(an + b)) g_{\hat{a}}(\ell) \mathbf{e}(\ell(\hat{a}(n + h) + \hat{b})) f_h(n) \\ &= \sum_{|k|, |\ell| \leq K} g_a(k) \mathbf{e}(kb) g_{\hat{a}}(\ell) \mathbf{e}(\ell \hat{b}) \cdot \mathbf{e}(\ell \hat{a} h) \sum_{n \leq x} \mathbf{e}((ka + \ell \hat{a})n) f_h(n). \end{aligned}$$

Therefore

$$\begin{aligned} \pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) &= \sum_{|k|, |\ell| \leq K} g_a(k) \mathbf{e}(kb) g_{\hat{a}}(\ell) \mathbf{e}(\ell \hat{b}) \cdot \mathbf{e}(\ell \hat{a} h) \int_{3^-}^x \mathbf{e}((ka + \ell \hat{a})u) d(S_h(u)) \\ &\quad + O(\Delta x + x^{1-1/\tau+\varepsilon} + K^{-1}\Delta^{-1}x), \end{aligned} \quad (3.7)$$

which completes our initial goal of expressing $\pi_h(x; \mathcal{B}, \hat{\mathcal{B}})$ in terms of the function S_h . To proceed further, it is useful to recall certain aspects of the analysis of S_h

that is carried out in [3]. First, writing $\tilde{\mathbf{1}}_{\mathbb{P}}(n) := \mathbf{1}_{\mathbb{P}}(n) - 1/\log n$, up to an error term of size $O(x^{1/2+\varepsilon})$ the quantity $S_h(x)$ is equal to

$$\begin{aligned} & \sum_{n \leq x} \left(\tilde{\mathbf{1}}_{\mathbb{P}}(n) + \frac{1}{\log n} \right) \left(\tilde{\mathbf{1}}_{\mathbb{P}}(n+h) + \frac{1}{\log n} \right) \prod_{0 < t < h} \left(1 - \frac{1}{\log n} - \tilde{\mathbf{1}}_{\mathbb{P}}(n+t) \right) \\ &= \sum_{\mathcal{A} \subseteq \{0, h\}} \sum_{\mathcal{T} \subseteq [1, h-1]} (-1)^{|\mathcal{T}|} \sum_{n \leq x} \left(\frac{1}{\log n} \right)^{2-|\mathcal{A}|} \left(1 - \frac{1}{\log n} \right)^{h-1-|\mathcal{T}|} \prod_{t \in \mathcal{A} \cup \mathcal{T}} \tilde{\mathbf{1}}_{\mathbb{P}}(n+t); \end{aligned}$$

see [3, Equations (2.5) and (2.6)]. By the modified Hardy-Littlewood conjecture (2.3) the estimate

$$\begin{aligned} \sum_{n \leq x} (\log n)^{-c} \prod_{t \in \mathcal{H}} \tilde{\mathbf{1}}_{\mathbb{P}}(n+t) &= \int_{3^-}^x (\log u)^{-c} d \left(\sum_{n \leq u} \prod_{t \in \mathcal{H}} \tilde{\mathbf{1}}_{\mathbb{P}}(n+t) \right) \\ &= \mathfrak{S}_0(\mathcal{H}) \int_3^x (\log u)^{-c-|\mathcal{H}|} du + O(x^{1/2+\varepsilon}) \end{aligned}$$

holds uniformly for any constant $c > 0$; consequently, up to an error term of size $O(x^{1/2+\varepsilon})$ the quantity $S_h(x)$ is equal to

$$\sum_{\mathcal{A} \subseteq \{0, h\}} \sum_{\mathcal{T} \subseteq [1, h-1]} (-1)^{|\mathcal{T}|} \mathfrak{S}_0(\mathcal{A} \cup \mathcal{T}) \int_3^x (\log u)^{-2-|\mathcal{T}|} \nu(u)^{h-1-|\mathcal{T}|} du,$$

where

$$\nu(u) := 1 - \frac{1}{\log u} \quad (u > 1)$$

(note that $\nu(u)$ is the same as $\alpha(u)$ in the notation of [3]). For every integer $L \geq 0$ we denote

$$\mathcal{D}_{h,L}(u) := \sum_{\substack{\mathcal{A} \subseteq \{0, h\} \\ (|\mathcal{A}| + |\mathcal{T}| = L)}} \sum_{\mathcal{T} \subseteq [1, h-1]} (-1)^{|\mathcal{T}|} \mathfrak{S}_0(\mathcal{A} \cup \mathcal{T}) (\nu(u) \log u)^{-|\mathcal{T}|} \nu(u)^h,$$

so that

$$S_h(x) = \sum_{L=0}^{h+1} \int_3^x \nu(u)^{-1} (\log u)^{-2} \mathcal{D}_{h,L}(u) du + O(x^{1/2+\varepsilon}).$$

We now combine this relation with (3.7), sum over the even natural numbers $h \leq (\log x)^3$, and apply (3.1) to deduce that the quantity $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ is equal to

$$\sum_{\substack{h \leq (\log x)^3 \\ 2|h}} \sum_{L=0}^{h+1} \sum_{|k|, |\ell| \leq K} g_a(k) \mathbf{e}(kb) g_{\hat{a}}(\ell) \mathbf{e}(\ell \hat{b}) \cdot \mathbf{e}(\ell \hat{a} h) \int_3^x \frac{\mathbf{e}((ka + \ell \hat{a})u)}{\nu(u) (\log u)^2} \mathcal{D}_{h,L}(u) du$$

up to an error term of size

$$\ll \frac{x}{(\log x)^3} + (\Delta x + x^{1-1/\tau+\varepsilon} + K^{-1} \Delta^{-1} x) (\log x)^3.$$

Choosing $\Delta := (\log x)^{-6}$ and $K := (\log x)^{12}$ the combined error is $O(x/(\log x)^3)$, which is acceptable.

Next, arguing as in [3] and noting that

$$\sum_{|k|, |\ell| \leq K} |g_a(k)g_{\hat{a}}(\ell)| \ll (\log \log x)^2,$$

one sees that the contribution to $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ coming from terms with $L \geq 3$ does not exceed $O(x/(\log x)^{5/2})$. Since $\mathcal{D}_{h,1}$ is identically zero (as \mathfrak{S}_0 vanishes on singleton sets), this leaves only the terms with $L = 0$ or $L = 2$. The function $\mathcal{D}_{h,2}$ splits naturally into four pieces according to whether $\mathcal{A} = \emptyset, \{0\}, \{h\}$ or $\{0, h\}$. Consequently, up to $O(x/(\log x)^{5/2})$ we can express the quantity $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ as

$$\sum_{j=1}^5 \sum_{|k|, |\ell| \leq K} g_a(k) \mathbf{e}(kb) g_{\hat{a}}(\ell) \mathbf{e}(\ell \hat{b}) \int_3^x \frac{\mathbf{e}((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} \mathcal{F}_{j,\ell}(u) du, \quad (3.8)$$

where (taking into account Lemma 2.3) we have written

$$\sum_{\substack{h \leq (\log x)^3 \\ 2|h}} \mathbf{e}(\ell \hat{a} h) \mathcal{D}_{h,L}(u) = \sum_{j=1}^5 \mathcal{F}_{j,\ell}(u) + O(x^{-1})$$

with

$$\begin{aligned} \mathcal{F}_{1,\ell}(u) &:= \sum_{\substack{h \geq 1 \\ 2|h}} \nu(u)^h \mathbf{e}(\ell \hat{a} h), \\ \mathcal{F}_{2,\ell}(u) &:= \sum_{\substack{h \geq 1 \\ 2|h}} \mathfrak{S}_0(\{0, h\}) \nu(u)^h \mathbf{e}(\ell \hat{a} h), \\ \mathcal{F}_{3,\ell}(u) &:= \frac{(-1)}{\nu(u) \log u} \sum_{\substack{h \geq 1 \\ 2|h}} \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{0, t\}) \nu(u)^h \mathbf{e}(\ell \hat{a} h), \\ \mathcal{F}_{4,\ell}(u) &:= \frac{(-1)}{\nu(u) \log u} \sum_{\substack{h \geq 1 \\ 2|h}} \sum_{1 \leq t \leq h-1} \mathfrak{S}_0(\{t, h\}) \nu(u)^h \mathbf{e}(\ell \hat{a} h), \\ \mathcal{F}_{5,\ell}(u) &:= \frac{1}{(\nu(u) \log u)^2} \sum_{\substack{h \geq 1 \\ 2|h}} \sum_{1 \leq t_1 < t_2 \leq h-1} \mathfrak{S}_0(\{t_1, t_2\}) \nu(u)^h \mathbf{e}(\ell \hat{a} h). \end{aligned}$$

First, we show that certain terms in (3.8) make a negligible contribution that does not exceed $O(x/(\log x)^{3/2-\varepsilon})$.

For any $\ell \neq 0$, using Lemma 2.4 with $\lambda = \ell \hat{a}$ we have

$$\mathcal{F}_{1,\ell}(u) = R_{0,0;\ell \hat{a}}(u) \ll \ell^{-4}$$

provided that $|\ell \hat{a}| \geq (\log u)^{-1}$, and for this it suffices that $u \geq \exp(\hat{a})$. Thus,

$$\int_3^x \frac{\mathbf{e}((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} \mathcal{F}_{1,\ell}(u) du \ll 1 + \ell^{-4} \frac{x}{(\log x)^2}.$$

In view of (3.3), the contribution to (3.8) from terms with $j = 1$ and $\ell \neq 0$ is

$$\ll \sum_{\substack{|k|, |\ell| \leq K \\ \ell \neq 0}} |g_a(k)| \cdot |\ell|^{-1} \left(1 + \ell^{-4} \frac{x}{(\log x)^2} \right) \ll \frac{x \log \log x}{(\log x)^2} \ll \frac{x}{(\log x)^{3/2-\varepsilon}}.$$

Similarly, for $\ell \neq 0$ and $u \geq \exp(\hat{\alpha})$ we have $\mathcal{F}_{2,\ell}(u) = S_{\ell\hat{a}}(u) \ll \ell^{-4}$ by Lemma 2.4, so the contribution to (3.8) from terms with $j = 2$ and $\ell \neq 0$ is also $O(x/(\log x)^{3/2-\varepsilon})$.

For any $\ell \in \mathbb{Z}$, by Lemma 2.2 and Lemma 2.4 we have

$$\max \{ |\mathcal{F}_{3,\ell}(u)|, |\mathcal{F}_{4,\ell}(u)| \} \ll \frac{1}{\log u} \sum_{\substack{h \geq 1 \\ 2|h}} h^{1/2+\varepsilon/2} \nu(u)^h \ll (\log u)^{1/2+\varepsilon/2},$$

hence for $j = 3, 4$ we see that

$$\int_3^x \frac{\mathbf{e}((ka + \ell\hat{a})u)}{\nu(u)(\log u)^2} \mathcal{F}_{j,\ell}(u) du \ll \frac{x}{(\log x)^{3/2-\varepsilon/2}}.$$

By (3.3), it follows that the contribution to (3.8) from terms with $j = 3, 4$ is

$$\ll \frac{x}{(\log x)^{3/2-\varepsilon/2}} \sum_{|k|, |\ell| \leq K} |g_a(k)g_{\hat{a}}(\ell)| \ll \frac{x(\log \log x)^2}{(\log x)^{3/2-\varepsilon/2}} \ll \frac{x}{(\log x)^{3/2-\varepsilon}}.$$

Finally, for any $\ell \in \mathbb{Z}$ and $u \geq \exp(\hat{\alpha})$, by Lemma 2.2 and Lemma 2.4 we have

$$\begin{aligned} \mathcal{F}_{5,\ell}(u) &= \frac{1}{(\nu(u) \log u)^2} \sum_{\substack{h \geq 1 \\ 2|h}} \left(-\frac{1}{2}h \log h + \frac{1}{2}Ah + O(h^{1/2+\varepsilon/2}) \right) \nu(u)^h \mathbf{e}(\ell\hat{a}h) \\ &= \frac{-\frac{1}{2}R_{1,1;\ell\hat{a}}(u) + \frac{1}{2}AR_{1,0;\ell\hat{a}}(u) + O(R_{1/2+\varepsilon/2,0;0}(u))}{(\nu(u) \log u)^2} \\ &\ll \frac{\lambda^{-4} + (\log u)^{3/2+\varepsilon/2}}{(\log u)^2}, \end{aligned}$$

and arguing as before we see that the contribution to (3.8) coming from terms with $j = 5$ does not exceed $O(x/(\log x)^{3/2-\varepsilon})$.

Applying the preceding bounds to (3.8) we see that, up to $O(x/(\log x)^{3/2-\varepsilon})$, the quantity $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ is equal to

$$\hat{a} \sum_{j=1,2} \sum_{|k| \leq K} g_a(k) \mathbf{e}(kb) \int_3^x \frac{\mathbf{e}(kau)}{\nu(u)(\log u)^2} \mathcal{F}_{j,0}(u) du,$$

where we have used the fact that $g_{\hat{a}}(0) = \hat{a}$. By Lemma 2.4 we have

$$\mathcal{F}_{1,0}(u) = \sum_{\substack{h \geq 1 \\ 2|h}} \nu(u)^h = R_{0,0;0}(u) = \frac{1}{2} \log u + O(1)$$

and

$$\mathcal{F}_{2,0}(u) = \sum_{\substack{h \geq 1 \\ 2|h}} \mathfrak{S}_0(\{0, h\}) \nu(u)^h = S_0(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1);$$

therefore,

$$\int_3^x \frac{\mathbf{e}(kau)}{\nu(u)(\log u)^2} \mathcal{F}_{j,0}(u) du = \frac{1}{2} \int_3^x \frac{\mathbf{e}(kau)}{\nu(u) \log u} du + O\left(\frac{x \log \log x}{(\log x)^2}\right) \quad (j = 1, 2).$$

Consequently, up to $O(x/(\log x)^{3/2-\varepsilon})$ we can express the quantity $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ as

$$\hat{a} \sum_{|k| \leq K} g_a(k) \mathbf{e}(kb) \int_3^x \frac{\mathbf{e}(kau)}{\nu(u) \log u} du. \quad (3.9)$$

To complete the proof of Theorem 1.1, we apply Lemma 2.5, which shows that the term $k = 0$ in (3.9) contributes

$$a\hat{a} \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = (\alpha\hat{\alpha})^{-1} \pi(x) + O\left(\frac{x}{(\log x)^2}\right)$$

to the quantity $\pi(x; \mathcal{B}, \hat{\mathcal{B}})$ (and thus accounts for the main term), whereas the terms in (3.9) with $k \neq 0$ contribute altogether only a bounded amount.

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